Gaussian LHL for Modules over Number Fields

Joint work with Martin R. Albrecht, Joël Felderhoff, Russell W. F. Lai, and Ivy K. Y. Woo

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Classic Leftover Hash Lemma (LHL)

$$(A, A \cdot x \bmod q) \sim (A, \mathcal{U}(\mathcal{R}_q^n))$$

where

$$A \leftarrow \mathcal{U}(\mathcal{R}_q^{n \times m}), \quad x \leftarrow \chi$$

and χ is usually a short distribution over $\mathcal{R}^m$ or $\mathcal{R}_q^m$

Gaussian LHL

$$(X, X \cdot v) \sim (X, \mathcal{D}_{\mathcal{R}^r, \sigma})$$

where

$$X \leftarrow \left(\mathcal{D}_{\mathcal{R}^r, \mathbf{S}_x}\right)^m, \quad v \leftarrow \mathcal{D}_{\mathcal{R}^m, \mathbf{S}_v}$$

Notations

- Number field $\mathscr{K}$ of degree d.
- Field discriminant $\Delta_{\mathscr{K}}$.
- The ring of integers $\mathscr{R} = \mathscr{O}_{\mathscr{K}}$.
- We assume the canonical embedding of $\mathscr{R}$ has a basis $||\mathbf{B}_{\mathscr{R}}||_{\infty} \leq \delta_{\mathscr{K}}$.
- We assume $\mathscr{K}$ contains $\mathbb{Q}(\zeta_f)$ for some $f \geq 2$.

Gaussian Linear Transform

$$(X, X \cdot v) \sim \left(X, \mathscr{D}_{\sqrt{\mathbf{X} \cdot \mathbf{S}_v \cdot \mathbf{S}_v^T \cdot \mathbf{X}^T}}\right)$$

where

$$X \leftarrow \left(\mathscr{D}_{\mathbf{S}_x}\right)^m, \qquad v \leftarrow \mathscr{D}_{\mathbf{S}_v}$$

Discrete Gaussian version

$$(\mathbf{X}, \mathbf{X} \cdot \mathbf{v}) \sim (\mathbf{X}, \mathcal{D}_{\mathbf{X} \cdot \mathcal{R}^m, \sqrt{\mathbf{X} \cdot \mathbf{S}_v \cdot \mathbf{S}_v^T \cdot \mathbf{X}^T}})$$

for $\quad \mathbf{X} \leftarrow \left(\mathcal{D}_{\mathcal{R}^r, \mathbf{S}_x}\right)^m, \quad \mathbf{v} \leftarrow \mathcal{D}_{\mathcal{R}^m, \mathbf{S}_v}$

as long as $\quad \mathbf{S}_v \geq \eta_\varepsilon(\Lambda^\perp(\mathbf{X}))$

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for $\mathbf{X} \leftarrow \left(\mathcal{D}_{\mathcal{R}^r, \mathbf{S}_x}\right)^m, \quad \mathbf{v} \leftarrow \mathcal{D}_{\mathcal{R}^m, \mathbf{S}_v}$

as long as $\mathbf{S}_v \geq \eta_\varepsilon(\Lambda^\perp(\mathbf{X}))$

we also consider $\mathbf{X} \cdot \mathcal{R}^m = \mathcal{R}^r$, and $\sqrt{\mathbf{X} \cdot \mathbf{S}_v \cdot \mathbf{S}_v^T \cdot \mathbf{X}^T} \approx \sigma \cdot I$

Surjective Gaussian Matrices

Extending [NP20] to the ring setting

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- We reduce mod every prime ideal $\boldsymbol{p}$ and prove non-zero determinant w.h.p.

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- Number of prime ideals in $\mathcal{O}_{\mathscr{K}}$ = infinite.
- Only consider small prime ideals that could divide $\det(\mathbf{X}_0)$.
- Bound the number for $N(p) \leq B$ by $B/\log(B)$ (uses GRH).

The result

$$\Pr\left(\mathbf{X} \leftarrow (\mathcal{D}_{\mathcal{R}^r,\mathbf{s}_x})^m \text{ is surjective}\right) \geq 1 - 2^{-\lambda}$$

when $m \geq 2r + \dfrac{\lambda}{\log(N_{\mathcal{R}})}$, $N_{\mathcal{R}}$ - the norm of the smallest ideal

Short Kernel Basis

Lifting [AR16] to the ring setting

Short Kernel Basis

From the the kernel to the unit vector preimages

- Find short $\mathbf{U} \in \mathscr{R}^{m \times r}$ such that $\mathbf{X} \cdot \mathbf{U} = \mathbf{I}_r$
- Then for $\mathbf{I}_m - \mathbf{U} \cdot \mathbf{X}$ we have $\mathbf{X} \cdot (\mathbf{I}_m - \mathbf{U} \cdot \mathbf{X}) = \mathbf{X} - \mathbf{I}_r \cdot \mathbf{X} = \mathbf{0}$

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- Similarly if $\mathbf{X} \cdot \mathbf{U} = f \cdot \mathbf{I}_r$
- Then for $f \cdot \mathbf{I}_m - \mathbf{U} \cdot \mathbf{X}$ (still short) we have $\mathbf{X} \cdot (f \cdot \mathbf{I}_m - \mathbf{U} \cdot \mathbf{X}) = \mathbf{X} \cdot f - f \cdot \mathbf{I}_r \cdot \mathbf{X} = \mathbf{0}$

Short Kernel Basis

Integer case

For $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_m]$ with $||\mathbf{x}_i||_\infty \leq B$ consider the set $S = \sum_{i=1}^{j} \{0,1\} \cdot \mathbf{x}_i$

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As j grows the norms in S grow slower than cardinality.

For $\mathbf{v} \in S$: $||\mathbf{v}||_\infty \leq j \cdot B$ this is a set of size $(2jB + 1)^r$ but $|S| = 2^j$.

When $2^j \geq (2jB + 1)^r$ (or $j \geq 2r\log(Br)$) we have a collision or $\sum_{i=1}^{j} \{0, \pm 1\} \cdot \mathbf{x}_i = 0$

Short Kernel Basis

Sets with a shift

Consider the set $S_j = \sum_{i=1}^{j} \{0, \pm 1\} \cdot \mathbf{x}_i$

We define random variables:

- $\text{Win}_j : S_j \cap S_j + \mathbf{e}_1 \neq \emptyset$
- $\text{Gain}_j : \mathbf{x}_{j+1} \notin S_j \wedge ||\mathbf{x}_{j+1}||_\infty \leq \sigma_x \sqrt{m} = B$

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If $\text{Gain}_j = 1$ for $j \in [1, 2r\log(Br)]$ then $\sum_{j=1}^{2r\log(Br)} \{0, \pm 1\} \cdot \mathbf{x}_j = 0$ and $\pm \mathbf{x}_{\text{last}} = \sum \{0, \pm 1\} \cdot \mathbf{x}_j$.

Short Kernel Basis

We Gain often enough

When $\neg\textsf{Win}$ so $S_j \cap S_j + \mathbf{e}_1 = \varnothing$ we have $\rho_{\sigma_x}(S_j) + \rho_{\sigma_x}(S_j + \mathbf{e}_1) \leq \rho_{\sigma_x}(\mathbb{Z}^m)$

And also $\rho_{\sigma_x}(S_j) \approx_\delta \rho_{\sigma_x}(S_j + \mathbf{e}_1)$

Hence $\Pr(S_j) \leq \frac{1}{2} + \delta$ and $\Pr(\neg S_j) \geq \frac{1}{2} - \delta$

Short Kernel Basis

Issues with integers

For $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_m]$ with $||\mathbf{x}_i||_\infty \leq B$ consider the set $S = \sum_{i=1}^{j} \{0,1\} \cdot \mathbf{x}_i$

As j grows the norms in S grow slower than cardinality.

For $\mathbf{v} \in S$: $||\mathbf{v}||_\infty \leq j \cdot B$ this is a set of size $(2jB+1)^r$ but $|S| = 2^j$.

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Short Kernel Basis

We change the set

Consider the set $A = \mathbf{B}_{\mathscr{R}} \cdot \{0,1\}^d$ and $S_j = \{\sum_{i=1}^{j} a_i \cdot \mathbf{x}_i \mid a_i \in A\}$.

Now $|S_j| = 2^{dj}$ with norm $\mathbf{v} \in S_j : ||\mathbf{v}||_\infty \leq jB \cdot d \cdot \delta_{\mathscr{K}}$

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Consider the set $A = \mathbf{B}_{\mathscr{R}} \cdot \{0,1\}^d$ and $S_j = \{ \sum_{i=1}^{j} a_i \cdot \mathbf{x}_i \mid a_i \in A \}$.

Now $|S_j| = 2^{dj}$ with norm $\mathbf{v} \in S_j : ||\mathbf{v}||_\infty \leq jB \cdot d \cdot \delta_{\mathscr{K}}$

Pigeonhole gives $dr \cdot \log(2jB \cdot d\delta_{\mathscr{K}} + 1) \leq dj \cdot \log(2)$ so $m \approx r \log(rB \cdot d\delta_{\mathscr{K}})$

** we actually take $A = \mathbf{B}_{\mathscr{R}} \cdot \{0, \pm 1\}^d$

Short Kernel Basis

The proof doesn't work

Now for $S_j = \sum_{i=1}^{j} A \cdot \mathbf{x}_i$

We define random variables:

- $\text{Win}_j : S_j \cap S_j + \mathbf{e}_1 \neq \varnothing$
- $\text{Gain}_j : \mathbf{x}_{j+1} \notin S_i \wedge ||\mathbf{x}_{j+1}||_\infty \leq \sigma_x \sqrt{m} = B$

If $\text{Gain}_j = 1$ for $j \in [1, \approx r\log(Br \cdot d\delta_{\mathcal{K}})]$ then $\sum_{j=1}^{\approx r\log(Brd\delta_{\mathcal{K}})} a_j \cdot \mathbf{x}_j = 0$ and $a_{\text{last}} \cdot \mathbf{x}_{\text{last}} = \sum a_j \cdot \mathbf{x}_j$.

Short Kernel Basis

Change further

Consider two sets $S_j = \sum_{i=1}^{j} A \cdot \mathbf{x}_i$ and $\hat{S}_j = \{s/a \mid s \in S_j, a \in A \backslash \{0\}\}$

We define random variables:

- $\text{Win}_j : \hat{S}_j \cap \hat{S}_j + \mathbf{e}_1 \neq \emptyset$
- $\text{Gain}_j : \mathbf{x}_{j+1} \notin \hat{S}_j \wedge ||\mathbf{x}_{j+1}||_\infty \leq \sigma_x \sqrt{m} = B$ implies $\mathbf{x}_{j+1} \notin S_j \subset \hat{S}_j$

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Short Kernel Basis

One more improvement

Consider two sets $S_j = \sum_{i=1}^{j} A \cdot \mathbf{x}_i$ and $\hat{S}_j = \{s/a \mid s \in S_j, a \in A \backslash \{0\}\}$

We define random variables:

- $\text{Win}_j : \hat{S}_j + \zeta^x \cdot \mathbf{e}_1 \cap \hat{S}_j + \zeta^y \cdot \mathbf{e}_1 \neq \emptyset$ now $\Pr(\neg \hat{S}_j) \geq 1/f - \delta$
- $\text{Gain}_j : \mathbf{x}_{j+1} \notin \hat{S}_j \wedge || \mathbf{x}_{j+1} ||_\infty \leq \sigma_x \sqrt{m} = B$ implies $\mathbf{x}_{j+1} \notin S_j \subset \hat{S}_j$

If $\text{Gain}_j = 1$ for $j \in [1, \approx r \log(Br \cdot d\delta_{\mathcal{K}})]$ then $\sum_{j=1}^{\approx r \log(Brd\delta_{\mathcal{K}})} a_j \cdot \mathbf{x}_j = 0$ and $a_{\text{last}} \cdot \mathbf{x}_{\text{last}} = \sum a_j \cdot \mathbf{x}_j$.

Short Kernel Basis

Win condition

$\hat{S}_j + \zeta^x \cdot \mathbf{e}_1 \cap \hat{S}_j + \zeta^y \cdot \mathbf{e}_1 \neq \varnothing$ implies

$$s_1/a_1 - s_2/a_2 = (\zeta^y - \zeta^x) \cdot \mathbf{e}_1 = \zeta^y \cdot (1 - \zeta^{x-y}) \cdot \mathbf{e}_1$$

Short Kernel Basis

Win condition

$\hat{S}_j + \zeta^x \cdot \mathbf{e}_1 \cap \hat{S}_j + \zeta^y \cdot \mathbf{e}_1 \neq \emptyset$ implies

$$s_1/a_1 - s_2/a_2 = (\zeta^y - \zeta^x) \cdot \mathbf{e}_1 = \zeta^y \cdot (1 - \zeta^{x-y}) \cdot \mathbf{e}_1$$

$$u \cdot (s_1 \cdot a_2 - s_2 \cdot a_1) = a_1 a_2 \cdot f \cdot \mathbf{e}_1$$

where $u = \zeta^{f-y} \cdot \dfrac{f}{1 - \zeta^{x-y}} \in \mathscr{R}$ is short

However for every $\mathbf{e}_i$ the value $a_1 a_2$ may be different.

Short Kernel Basis

We run it twice

Define $B = \{b \text{ s.t. } ||b||_\infty \leq R,$ coprime with $a_1 a_2\}$

We prove there exists $|B| \geq 2^d$ for $R = O(\Delta_{\mathcal{K}}^{1/d} \cdot d^{2.5} \cdot \delta_{\mathcal{K}}^3)$.

We get $u' \cdot (s_1' \cdot b_2 - s_2' \cdot b_1) = b_1 b_2 \cdot f \cdot \mathbf{e}_1$ using small Bezout identity $\alpha \cdot a_1 a_2 + \beta \cdot b_1 b_2 = 1$

We finally get $\alpha \cdot u \cdot (s_1 \cdot a_2 - s_2 \cdot a_1) + \beta \cdot u' \cdot (s_1' \cdot b_2 - s_2' \cdot b_1) = f \cdot \mathbf{e}_1$

Short Kernel Basis

The result and open problems

$$\Pr\left(\eta_\varepsilon(\Lambda^\perp(\mathbf{X})) \geq O(m^{3/2} \cdot d^{12} \cdot f^2 \cdot \delta_{\mathcal{K}}^{14} \cdot \Delta_{\mathcal{K}}^{4/d} \cdot s_{\min}(\mathbf{S}_x))\right) \geq 1 - 2^{-\lambda}$$

when $m \geq r \log(d \cdot r \cdot s_{\max}(\mathbf{S}_x))^{1+o(1)} + \dfrac{\lambda}{\log f}$.

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The result and open problems

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when $m \geq r \log(d \cdot r \cdot s_{\max}(\mathbf{S}_x))^{1+o(1)} + \dfrac{\lambda}{\log f}$.

- Find a different set A that contains:
  - Short unit elements, with short inverses.
- Find a set A where all elements have many coprime short values.

Spherical covariance

Spherical covariance

First approach

Set $\mathbf{S}_v$ as pseudo-inverse of $\mathbf{X}$ scaled by σ. Then $\sqrt{\mathbf{X} \cdot \mathbf{S}_v \cdot \mathbf{S}_v^T \cdot \mathbf{X}^T} = \sigma \cdot I$

And $$\frac{\sigma}{s_{\max}(\mathbf{X})} \leq s_{\min}(\mathbf{S}_v) \leq s_{\max}(\mathbf{S}_v) \leq \frac{\sigma}{s_{\min}(\mathbf{X})}$$

Spherical covariance

Second approach

- [AGHS13] conjectured that singular values of $\sqrt{\mathbf{X} \cdot \mathbf{X}^T}$ are close for a large enough m
- We refine their bound and apply it to rings

Spherical covariance

Rotating a Real Gaussian Matrix [Sil85]

Assume $\mathbf{Z} \leftarrow (\mathscr{D}_{\sigma=1})^{r\times m}$ then there exist orthogonal $\mathbf{O}_i$ s.t.

$$\mathbf{Z} = \begin{bmatrix} — & \mathbf{z}_1^{\mathrm{T}} & — \\ — & \mathbf{z}_2^{\mathrm{T}} & — \\ & \cdots & \end{bmatrix}$$

$$\mathbf{Z}\cdot\mathbf{O}_0 = \begin{bmatrix} X_m & 0 \dots 0 \\ & \tilde{\mathbf{Z}}_0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \dots 0 \\ 0 & \\ \vdots & \tilde{\mathbf{O}}_1 \\ 0 & \end{bmatrix} \cdot \mathbf{Z}\cdot\mathbf{O}_0 = \begin{bmatrix} X_m & 0 \dots 0 \\ Y_{r-1} & \\ \vdots & \tilde{\mathbf{Z}}_1 \\ 0 & \end{bmatrix}$$

where X_m, Y_{r-1} are Chi random variables of corresponding dimension.

Spherical covariance

Gershgorin circle theorem

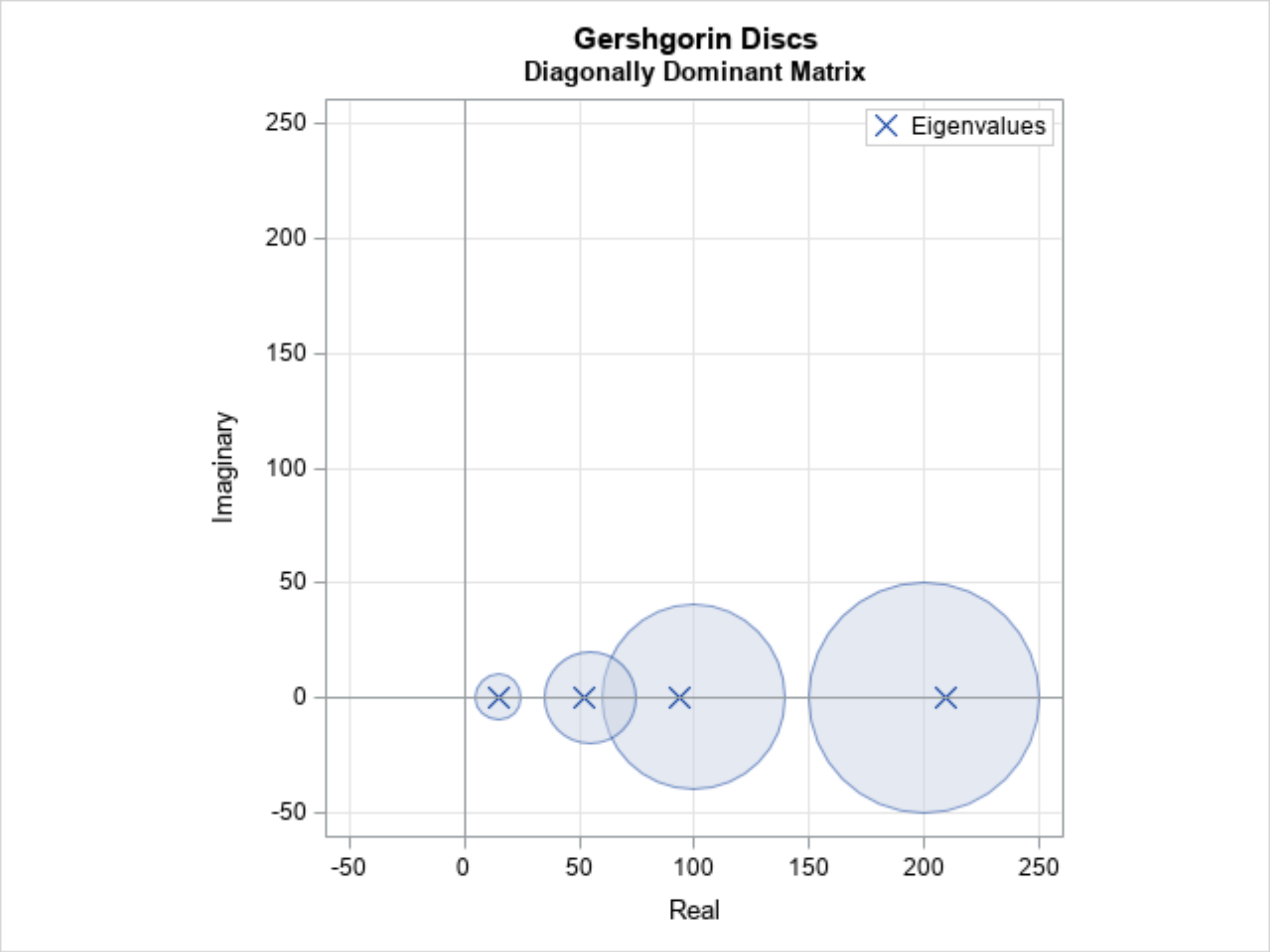
$$\mathbf{O} \cdot \mathbf{Z} \cdot \mathbf{Z}^{\mathrm{T}} \cdot \mathbf{O}^{\mathrm{T}} =$$

$$\begin{bmatrix} X_m^2 & X_m Y_{r-1} & 0 & & & 0 \\ X_m Y_{r-1} & X_{m-1}^2 + Y_{r-1}^2 & X_{m-1} Y_{r-2} & 0 & & 0 \\ & & \cdots & \cdots & & \\ 0 & & & 0 & X_{m-r+2} Y_1 & X_{m-r+1}^2 + Y_1^2 \end{bmatrix}$$

Then [Sil85] applies the Gershgorin circle theorem and Chi tail bounds

Spherical covariance

Gershgorin circle theorem



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Spherical covariance

Moving to Discreet Gaussians

$$\Pr\left(1 - \frac{1}{rd} \leq \lambda_{\min}/m \leq \lambda_{\max}/m \leq 1 + \frac{1}{rd}\right) \geq 1 - 4r \cdot \exp(-\lambda) \quad \text{for} \quad m \geq 49 \max(r, \lambda) \cdot (rd)^2$$

- We add continuos noise to embedding of $\mathbf{X}$ such that $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ is close to continuos
- Apply [Sil85] result
- We prove RD between this distribution and spherical Gaussian is a constant

Application to k-SIS

Gaussian matrix with a trapdoor from [LPSS14].

$\mathsf{HintTG}_{m,k}(\varsigma_1, \varsigma_2) \to (\mathbf{X} \in \mathcal{O}_{\mathcal{K}}^{k\times(m+k)}, \mathbf{U} \in \mathcal{O}_{\mathcal{K}}^{(m+k)\times(m+k)})$

1 : $\mathbf{X}_1 \leftarrow \mathcal{D}^m_{\mathcal{O}^k_{\mathcal{K}}, \varsigma_1}$

2 : $\mathbf{\Sigma} \coloneqq \sqrt{\mathbf{S}\cdot\mathbf{S}^\mathsf{T}}$ s.t. $\varsigma_2^2 \cdot \mathbf{I}_{dr} = \tilde{\Phi}(\mathbf{X}_1)\cdot\mathbf{S}\cdot\mathbf{S}^\mathsf{T}\cdot\tilde{\Phi}(\mathbf{X}_1)^\mathsf{T}$

3 : $\forall : i \in [k] : \mathbf{r}_i \leftarrow \mathcal{D}_{\mathcal{O}^m_{\mathcal{K}}, \sqrt{\mathbf{\Sigma}}}$. Let $\mathbf{R} \coloneqq (\mathbf{r}_1, \ldots, \mathbf{r}_k)$

4 : $\mathbf{X}_2 \coloneqq \mathbf{X}_1 \cdot \mathbf{R} + \mathbf{I}_k \in \mathcal{O}_{\mathcal{K}}^{k\times k}$

5 : $\mathbf{U} \coloneqq \begin{bmatrix} -\mathbf{R} & -\mathbf{I}_m - \mathbf{R}\mathbf{X}_1 \\ \mathbf{I}_k & \mathbf{X}_1 \end{bmatrix} \in \mathcal{O}_{\mathcal{K}}^{(m+k)\times(m+k)}$

6 : **return** $((\mathbf{X}_1, \mathbf{X}_2), \mathbf{U})$

Thank you!

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